ON PARAMETRIC EXCITATION OF CONVECTIVE INSTABILITY

(O PARAMETRICHESKOM VOZBUZHDENII KONVEKTIVNOI NEUSTOICHIVOSTI)

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Convective stability of fluid in a gravity field is usually investigated under the assumption that the equilibrium temperature gradient does not depend upon time. However, unsteady equilibrium of a fluid is also possible when the equilibrium temperature varies with time according to a law that is determined by the unsteady heating conditions. Investigation of the stability of such an unsteady equilibrium has, to our knowledge, not been carried out.

Among various possible unsteady equilibria, the most interesting is probably the case when the equilibrium temeprature gradient changes periodically with time. In this case the fluid is a singular oscillating system with a periodically varying parameter, and one can expect under such conditions the appearance of interesting phenomena of the type of parametric resonance.

We investigate below the stability of the equilibrium of a plane horizontal layer of fluid with a periodically varying temperature gradient. The solution of this case makes it possible to see clearly the characteristic singularity of the problem.

1. We consider a plane horizontal layer of fluid, bounded by the planes $z = \pm h$ (the z-axis being directed vertically). In equilibrium the velocity of the fluid $\mathbf{v} = 0$, and the equilibrium temperature $T_0 = T_0(z, t)$ satisfies the unsteady equation of heat conduction

$$\partial T_0 / \partial t = \chi \partial^2 T_0 / \partial z^2 \qquad (1.1)$$

where χ is the coefficient of heat conduction. We will consider heating conditions such that the equilibrium temperature gradient in the fluid varies periodically about some mean value with frequency ω_0. Consideration is restricted to the regime of low frequencies, satisfying the condition

$$\omega_0 \ll \chi / h^2 \tag{1.2}$$

(weak thermal skin effect). In this case the equilibrium temperature gradient does not depend upon z

$$\partial T_0 / \partial z = -A_0 + a_0 \varphi(t) \tag{1.3}$$

where $\varphi(t)$ is a modulating function with period $2\pi/\omega_0$, and the constants A_0 and a_0 represent the mean temperature gradient and its amplitude of modulation.

For the investigation of the stability of unsteady equilibrium we obtain from the equations of convection [1] the usual form of small-disturbance equations

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \triangle \mathbf{v} - \mathbf{g}\beta T \tag{1.4}$$

$$\frac{\partial T}{\partial t} + v_z \frac{\partial T_0}{\partial z} = \chi \triangle T, \qquad \operatorname{div} \mathbf{v} = 0 \tag{1.5}$$

In contrast with the small-disturbance equations of steady equilibrium, here one of the coefficients, $\partial T_0/\partial z$, depends upon time.

Considering periodic perturbations in the xy plane, we suppose that all quantities in equations (1.4) to (1.5) are proportional to $\exp\{i(k_1 x + k_2 y)\}$. Eliminating v_x, v_y and p, we obtain for the parts of the perturbation of temperature T and vertical velocity v_z depending upon z and t

$$\frac{\partial}{\partial t}(v_z'' - k^2 v_z) - \nu (v_z'''' - 2k^2 v_z'' + k^4 v_z) = -g\beta k^2 T \quad (k^2 = k_1^2 + k_2^2) \tag{1.6}$$

$$\frac{\partial T}{\partial t} + v_z T_0' = \chi (T'' - k^2 T) \tag{1.7}$$

Here primes denote differentiation with respect to z.

The simplest dependence of the perturbations upon z is obtained in the case when the layer is bounded by free layers (Rayleigh's case)

$$T = 0, \qquad v_z = v_z'' = 0 \qquad \text{at } z = \pm h \tag{1.8}$$

Then setting

$$v_z = v(t) \cos \frac{\pi}{2h} z, \qquad T = \tau(t) \cos \frac{\pi}{2h} z \tag{1.9}$$

we obtain, substituting (1.9) into (1.6) and (1.5), equations for the amplitudes $v(t)$ and $\tau(t)$

$$\dot{v} + \nu \varkappa^2 v = \frac{g\beta k^2}{\varkappa^2}\tau, \qquad \dot{\tau} + \chi\varkappa^2\tau = -T_0'v \qquad \left(\varkappa^2 = k^2 + \frac{\pi^2}{4h^2}\right) \tag{1.10}$$

Eliminating $\tau(t)$ from this system, we obtain the equation

$$\ddot{v} + \varkappa^2\chi\left(1 + \frac{\nu}{\chi}\right)\dot{v} + \left[\frac{g\beta k^2}{\varkappa^2}T_0' + \nu\chi\varkappa^4\right]v = 0 \tag{1.11}$$

Choosing the unit of time as $1/\kappa^2\sqrt{(\nu\chi)}$ and substituting T_0' from (1.3), we put this equation into the form

$$\ddot{v} + 2\varepsilon\dot{v} + [1 - R + r\varphi(t)]\,v = 0 \tag{1.12}$$

Here

$$R = \frac{g\beta A_0}{\nu\chi}\frac{k^2}{\varkappa^6}, \qquad r = \frac{g\beta a_0}{\nu\chi}\frac{k^2}{\varkappa^6}, \qquad 2\varepsilon = \frac{1+P}{\sqrt{P}}, \qquad P = \frac{\nu}{\chi}$$

The function $\varphi(t)$ in equation (1.12) has the period

$$T = \frac{2\pi}{p_*} \qquad \left(p_* = \frac{\omega_0}{\varkappa^2\sqrt{\nu\chi}}\right)$$

Here p_* is the dimensionless frequency of modulation.

Thus the behavior with time of the perturbations is governed by the Hill equation with a term representing damping.

2. If the parametric excitation is absent ($r = 0$), we obtain the well-known problem of Rayleigh [2] for the stability of equilibrium of a layer with free surfaces and a steady temperature gradient. In this case all coefficients of the equation for v are constant, and the solution depends upon time according to the law $\exp(\lambda t)$. For the decrement λ we find

$$\lambda_\pm = -\frac{1+P}{2\sqrt{P}} \pm \sqrt{\frac{(1-P)^2}{4P} + R} \tag{2.1}$$

As is evident from (2.1), for heating from above ($R < 0$) the real parts of λ_+ and λ_- are negative for all R (disturbances are damped), where for $|R| < 1/4(1 - P)^2/P$ the decrements λ_+ and λ_- are real (monotone damping) and for $|R| > 1/4(1 - P)^2/P$ the decrements λ_+ and λ_- are complex-conjugate (oscillatory damping).

For heating from below ($R > 0$) both decrements are always positive (monotone perturbations), with $\lambda_- < 0$ for all R, and λ_+ growing with increasing R and becoming positive at $R = 1$, which represents the limit of stability of stationary equilibrium.

In the presence of parametric excitation the problem is to determine the regions of stability and instability for the solution of equation (1.12) as they depend upon the values of the parameters R, r, p_* and ε.

The case of greatest interest is evidently sinusoidal modulation: $\varphi(t) = \sin p_* t$. In this case the boundaries of the region of stability can for small values of the excitation parameter r be found by means of the method of a small parameter [3]. Our system, however, possesses strong damping: the parameter ε as a function of the Prandtl number P has a minimum at $P = 1$ ($\nu = \chi$), where $\varepsilon_{min} = 1$. Therefore arbitrary (not small) values of the parameter r are of interest. In this regime the stability boundaries of the solution of equation (1.12) can be found more easily if the sinusoidal modulation is replaced by a rectangular one. As is well known [4,5], the general properties of the solution of Hill's equation are almost unchanged by such a substitution.

If the modulation follows a rectangular law (Fig. 1), we have on Sections 1 and 2 $\varphi = \mp 1$, and the general solution of equation (1.12)

$$v^{(1)} = e^{-\varepsilon t}(C_1 \sin \alpha t + C_2 \cos \alpha t), \qquad \alpha = \sqrt{1 - R - \varepsilon^2 - r} \tag{2.2}$$

$$v^{(2)} = e^{-\varepsilon t}(C_3 \sin \beta t + C_4 \cos \beta t), \qquad \beta = \sqrt{1 - R - \varepsilon^2 + r} \tag{2.3}$$

At $t = 0$, v and $\dot{v}$ must be continuous

$$v^{(1)}(0) = v^{(2)}(0), \qquad \dot{v}^{(1)}(0) = \dot{v}^{(2)}(0) \tag{2.4}$$

We will seek a periodic solution of equation (1.12). We therefore require fulfillment of the conditions of periodicity

$$v^{(2)}(\pi / p_*) = \pm v^{(1)}(-\pi / p_*), \qquad \dot{v}^{(2)}(\pi / p_*) = \pm \dot{v}^{(1)}(-\pi / p_*) \tag{2.5}$$

A solution satisfying (2.5) would represent steady oscillation with frequency

$$\Omega = np_* \tag{2.6}$$

where the plus sign in (2.5) corresponds to integral values of n and the minus sign to half-integral values.

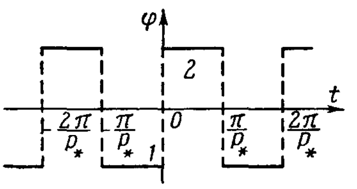

Fig. 1.

Conditions (2.4) and (2.5) give a system of four linear homogeneous equations for the constants C_i. This system has a nontrivial solution if its determinant vanishes. Thus we find the conditions under which a periodic solution of equation (1.12) is possible

$$\cos\frac{\alpha}{p}\cos\frac{\beta}{p} - \frac{\alpha^2+\beta^2}{2\alpha\beta}\sin\frac{\alpha}{p}\sin\frac{\beta}{p} = \pm\cosh\frac{2\varepsilon}{p} \qquad \left(p = \frac{p_*}{\pi}\right) \tag{2.7}$$

The relation (2.7) determines the stability boundaries for unsteady equilibrium. Stable equilibrium corresponds to values of the parameters R, r, p and ε for which the left side of (2.7) lies within the interval $(-\cosh(2\varepsilon/p), \cosh(2\varepsilon/p))$.

3. For fixed values of the damping coefficient ε and the parameter R, which determine the mean temperature gradient, equation (2.7) gives the relation between the amplitude r and the frequency p of excitation on the stability boundary. We give the results of numerical solution of equation (2.7).

We consider first the region $-\infty < R < 1$. In the absence of excitation ($r = 0$) such values of R correspond to stable equilibrium (arbitrary heating from above, or heating from below with a temperature gradient less than the critical). In the presence of excitation ($r \neq 0$)

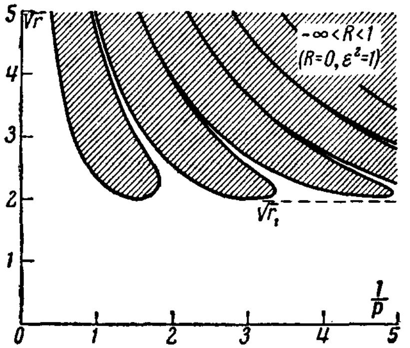

Fig. 2.

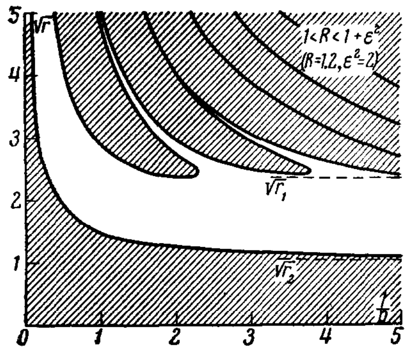

Fig. 3.

regions of instability appear. For example, Fig. 2 shows the first regions of instability for values of the parameters $R = 0$ and $\varepsilon = 1$, in the coordinates $(\sqrt{r}, p^{-1})$. For small r the equilibrium is stable for any frequency p. For fixed $r > r_1$, where r_1 is a certain threshold value, there appear, as seen in Fig. 2, intervals of frequencies corresponding to stability and instability (the regions of instability being cross-hatched). The intervals of stability contract with increasing r. The regions of instability show in Fig. 2 correspond to the integral and half-integral values $n = 1/2, 1, 3/2, 2, \ldots$ in equation (2.6). The equations of the lines separating adjacent regions of instability have, for large r, the form*

* The asymptotic behavior of the boundaries of the regions for large r does not depend upon the parameters R and ε and is always given by equation (3.1).

$$\frac{\sqrt{r}}{p} = (2l+1)\frac{\pi}{2} \qquad (l = 0, 1, 2, \ldots) \tag{3.1}$$

The threshold value of the excitation parameter is

$$r_1 = 3\varepsilon^2 - (R - 1) \tag{3.2}$$

We consider now the range $R > 1$. Under static conditions ($r = 0$) the equilibrium is unstable for $R > 1$. The modulating temperature gradient stabilizes the equilibrium for definite values of the frequency p and amplitude r of modulation. The form of the regions of stability and instability in the ($\sqrt{r}$, p^{-1}) plane are different in the cases $R \gtrless 1 + \varepsilon^2$.

Suppose that $1 < R < 1 + \varepsilon^2$. The form of the stability curves is evident from Fig. 3 ($R = 1.2$, $\varepsilon^2 = 2$). In contrast to the case $R < 1$, a region of instability now appears adjacent to the coordinate axes ($n = 1$ in equation (2.6)). The equation of the boundary of this region for $p^{-1} \to \infty$ is

$$r_2 = 2\varepsilon\sqrt{R - 1} \tag{3.3}$$

Above this region lies a strip of stability, whose width (for large p^{-1}) is equal to $r_1 - r_2$. As R increases the stability strip shrinks, since r_1 decreases and r_2 increases, and $r_1 - r_2 \to 0$ as $R \to 1 + \varepsilon^2$.

The form of the stability curves for $R > 1 + \varepsilon^2$ is evident from Fig.4 ($R = 4$, $\varepsilon^2 = 2$). In this case the equilibrium is unstable for practically all values of the frequency and amplitude of excitation. However, for $r > r_3$ there are narrow intervals of resonant frequency, for which parametric excitation leads to stabilization of the system (there is here a complete analogy with the behavior of an astatic pendulum with exciting support). The threshold value is

$$r_3 = \varepsilon^2 + (R - 1) \tag{3.4}$$

Figure 5 shows the threshold values r_1, r_2 and r_3 as functions of R. From (3.2) and (3.3) it is possible to determine the lowest critical value of the Rayleigh number R as a function of the amplitude of excitation r in the limiting case of low frequency ($p^{-1} \gg 1$). For $r < 2\varepsilon^2$ the critical value of R grows quadratically with increasing r; for $r > 2\varepsilon^2$ it decreases linearly

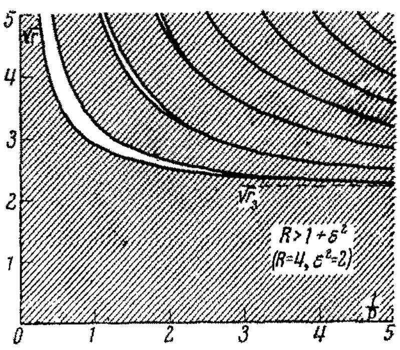

Fig. 4.

$$R = 1 + \frac{r^2}{4\varepsilon^2} \qquad (r < 2\varepsilon^2) \tag{3.5}$$

$$R = 1 + 3\varepsilon^2 - r \qquad (r > 2\varepsilon^2) \tag{3.6}$$

We give also the values of the critical number R in the limiting case of high frequency ($p^{-1} \ll 1$) and small amplitude of modulation ($r \ll 1$). Expanding the left and right side of (2.7) in powers of $1/p$ and r, we find

$$R = 1 + \frac{r^2}{12p^2} \tag{3.7}$$

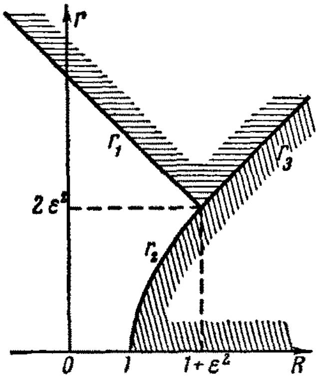

Fig. 5.

From the results obtained, it is evident that periodic modulation of the temperature gradient significantly affects the onset of convection. Stability or instability of the equilibrium is determined not only by the mean temperature gradient, but also depends in a complicated way upon the amplitude and frequency of modulation. Under particular conditions this dependence has a resonant character. Experimental investigation of convective stability would be of interest for unsteady equilibrium of a fluid, in particular in the case of periodic modulation of the equilibrium temperature gradient. In an experiment it would hardly be possible to realize in pure form the conditions that were adopted as simplifying assumptions in the present work: rectangular modulation, free surfaces, absence of skin effect. One might suppose, however, that the qualitative deductions would be insensitive to these assumptions.

We note in conclusion that an analogous effect should be observed also in the onset of hydrodynamic instability of a moving fluid. Thus in [6] the effect of modulation of the corner velocity upon the stability of fluid motion between rotating cylinders was observed experimentally.

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